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LETTERS TO THE EDITOR

PARAMETRIC ANALYSIS AND FRACTAL-LIKE BASINS OF ATTRACTION BY MODIFIED INTERPOLATED CELL MAPPING

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1. INTRODUCTION

It is well known that there may be multiple attracting stable motions, called attractors, in a non-linear dynamical system. The attractors and their basins of attraction in the region of interest must be delineated to characterize the global behavior of the system [1]. However, no analytical method can accomplish such a global analysis effectively. Recently, Ge and Lee developed a modified interpolated cell mapping method (MICM) for the global analysis of deterministic systems with smooth basin boundary [2].

In this paper, a method using MICM is first developed to locate all attractors of a system in a large region of study arbitrarily assigned, which is helpful for the global analysis of the systems never studied before. This method is then applied to the parametric analysis of a Duffing's system, in which the attracting cells located by MICM are further iterated forward by numerical integration to locate exact attractors. This parametric analysis is more complete and accurate than the analysis using ICM. Finally, MICM is used to determine fractal-like basins of attraction of dynamical systems. The basins of attraction located by global analysis using MICM are first used to examine whether the basin boundaries of a system are fractal. If a system possibly has fractal basin boundaries, each cell is divided into four small cells and the basins of attraction located by MICM are then reduced to positively invariant sets of small cells under sample mappings or under interpolated mappings. These sets are assigned to the attraction criteria for the attractors located by global analysis using MICM. Except for the points residing in these sets, all points are iterated forward by numerical integration to locate their attractors. For the three systems studied in this paper, the fractal-like basins of attraction located by the proposed method are almost identical to those located by IGP, with a 10-fold improvement in computational efficiency. Although ICM is more computationally efficient than MICM, the attractors of 10% of the total points are incorrectly located in the region of interest; in addition, the error increases in the analysis of fractal regions. However, MICM still has satisfactory accuracy in the studies of fractal regions.

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2. THE MICM METHOD

The MICM method was initially developed for the global analysis of dynamical systems with continuous basins of attraction separated by smooth basin boundary [2]. In global analysis using MICM, three kinds of cells are defined as below. Here the attracting, basin, and boundary cells are used to indicate the attractors, basins of attraction, and basin boundaries of the system.

Definition 1. The cells in which an attractor resides are called *the attracting cells of the attractor.*

Definition 2. If a cell and its adjoining cells [3] all lead to an identical attractor, the cell is called *a basin cell of the attractor.*

Definition 3. If a cell and one of its adjoining cells lead to different attractors, the cell is called *a boundary cell.*

Here, we summarize the algorithm of global analysis using MICM [2]. The region of interest in state space is divided into an array of cells. The reference mappings of interpolation and the sink cell are defined as in ICM. Then, the interpolated mapping sequence of each cell within N_p periods are constructed and recorded. From the mapping sequences of cells, periodic attractors with periods of less than N_p are located by a criterion of 10^{-5} cell size, and their basins of attraction are located by a criterion of 10^{-2} cell size. Except for the cells in the basins of attraction of the sink cell and the periodic attractors located, all cells are considered in the basin of attraction of an "undetermined attractor". The N_pth mappings of these cells are then iterated forward to construct the next interpolated mapping sequences. In addition, if a mapping of a cell reaches a basin cell of a periodic attractor located, the cell is considered in the basin of attraction of the periodic attractor. In such a construction of iterated mapping sequences, part of the cells studied will lead to the located periodic attractors after long enough periods. The construction of mappings of cells is finally halted upon satisfaction of two conditions. First, the number of cells attracted by the undetermined attractor does not decrease in an iteration of N_p periods. This indicates that the cells attracted by the periodic attractors located and by the sink cell are all located. Second, the number of attracting cells of the undetermined attractor does not decrease in the iteration, where the attracting cells within analysis are alternatively defined as the cells in which the N_pth mappings of cells attracted by the undetermined attractor reside. This indicates that the attracting cells of the undetermined attractor are located. However, the undetermined attractor is just the set of attractors not located by a 10^{-5} cell size. If multiple attractors belong to the undetermined attractor, the attractors and their basins of attraction must be further located for a complete global analysis. Hence, we assign all mappings of each attracting cell in the following iterations of N_p periods to the sample mappings of the attracting cell and include all the cells reached by the sample mappings into a set. In such a manner, MICM can divide the attracting cells of the undetermined attractor into different sets of attracting cells with each set indicating an attractor. Besides, if the final interpolated mapping of a cell leads to a set of attracting cells, the cell is considered in the basin of attraction of the set. Thus, MICM locates all attractors and their basins of attraction.

Since ICM [4] cannot distinguish multiple strange attractors or limit cycles, MICM can provide a more complete global analysis of dynamical systems than ICM. By constructing interpolated mapping sequences through iterative N_p periods, MICM can locate more complete basins of attraction of periodic attractors than ICM. However, the attractors and basins of attraction located by MICM are still not precise and complete enough due to the use of interpolated mappings and the basin of attraction of the sink cell in which the attractors of cells are still unknown. A method using "exact" mappings from numerical integration was developed for precise and complete global analysis [2]. To locate exact basins of attraction, the mappings of cells in the basin of attraction of the sink cell are first constructed by numerical integration until the cells reach the basin cells to identify their attractors as above. Then the basin and boundary cells of attractors are defined again since the attractors of all cells in the region of interest have been located. In addition, N_s^n interior-and-boundary sampling points [3] in each boundary cell are iterated forward by numerical integration until the points are mapped to the new basin cells to identify their attractors. In such a manner, the attractors of points in the region of interest are all located. Since this method uses exact mappings of points to locate attractors of points, the global analysis is complete and precise in comparison with the analysis using IGP. Besides, this method assigns the basin cells to the attraction criteria of attractors; thus, computational efficiency of global analysis using cell mapping methods is preserved. The method using exact mappings can also be used to locate exact attractors as follows. N_s^n interior-and-boundary points are uniformly sampled within each of the attracting cells located by MICM global analysis. Each sampling point is iterated forward by numerical integration to locate exact attractors.

3. GLOBAL ANALYSIS OF A LARGE REGION USING MICM

When global behavior of a dynamical system is studied by a cell mapping method with a fixed number of cells, selecting an appropriate region of study is important. If the region is large, a global analysis is less accurate due to the use of a larger cell size. If the region does not include an attractor wholly, the attractor cannot be located. However, before global analysis of a system that was never studied before, there is no available information about what the attractors are and where they reside. Hence, there is no easy way to judge the appropriateness of the region of study selected. To remedy this drawback of global analysis using cell mapping methods, a method using MICM is proposed to locate all attractors in a large region of study that is arbitrarily assigned. The attractors located can provide helpful information on the selection of an appropriate region of study for further analysis.

The proposed method is applied to the global analysis of a forced Duffing's oscillator, that was also studied in references [4, 5], governed by the following equation:

$$\ddot{x} + 0{\cdot}25\dot{x} + 0{\cdot}02x + x^3 = 8{\cdot}5\cos(t). \tag{1}$$

The point (0·1, 0·1) is taken to locate the first attractor. The strange attractor is located in the first attractive region, $(2{\cdot}19, 3{\cdot}36)\times(-1{\cdot}54, 3{\cdot}94)$. The periodic attractor is then located in another attractive region, $(2{\cdot}04, 3{\cdot}13)\times(-2{\cdot}67, 5{\cdot}34)$, that are defined by the steady state mappings of a cell in the basin of attraction of the sink cell of the first attractive region. Since these two attractive regions overlap, they are combined into a large attractive region $(2{\cdot}04, 3{\cdot}36)\times(-2{\cdot}67, 5{\cdot}34)$, that is still denoted as the first attractive region. After studying the basin of attraction of the sink cell, the attractors and basins of attraction in the first attractive region are as shown in Figure 1(a). The symbols " × " and " • " denote the basins of attraction of the periodic attractor and the strange attractor.

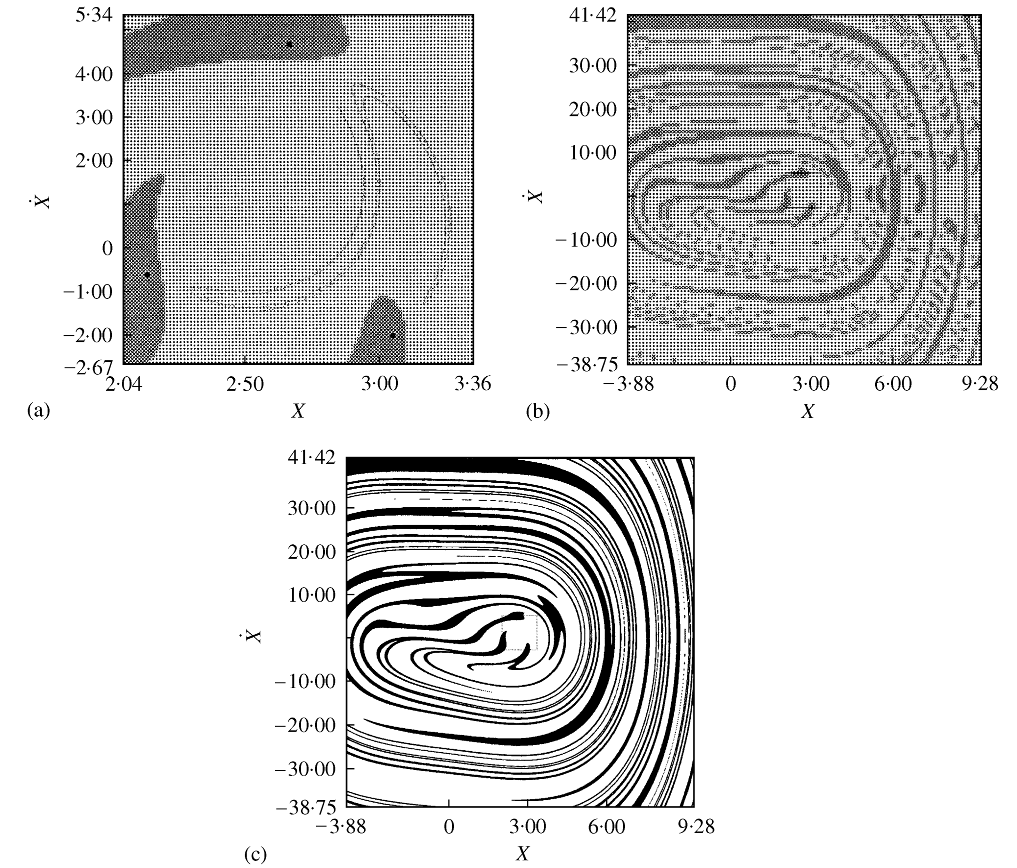

Figure 1. Global analysis of system (1) in a large region of study using the proposed method: (a) basins of attraction in the first attractive region; (b) basins of attraction in the large region of study; (c) detailed basins of attraction in the large region of study.

The first attractive region is further expanded by a factor 10, and which is assigned to the large region of study. No new attractor is located in the large region. The basins of attraction located by the proposed method are as shown in Figure 1(b). The detailed structure of the large region of study is also interesting. The attractors of 909^2 points are hence located, as shown in Figure 1(c). On average, these points require 3·38 periods to reach the basin cells located to identify their attractors. The basins of attraction located by the proposed method are the same as those located by IGP. In global analysis using IGP, the criteria 10^{-6} and 10^{-4} are assigned to locate periodic attractors and their basins of attraction; in addition, a point is assumed to be attracted by the strange attractor if the point cannot lead to a periodic attractor before 100 periods. On average, in IGP analysis the points require 63·88 periods to identify their attractors for this example. The proposed method hence has a 19-fold improvement in computational efficiency over IGP for global analysis of this system. In this paper, all computations were processed on an IRIS Indigo workstation with 16 MB main memory. The algorithm of the fourth order Runge–Kutta method is applied to numerical integration of initial conditions. In addition, the mapping time step duration, namely one period, in each of the periodic system [3] studied is divided into 100 integration steps for numerical integration.

Since the error of global analysis using a cell mapping method increases as a larger cell size is used, each cell mapping method cannot be directly applied to the global analysis of a large region. The proposed method uses MICM to study each small attractive region; therefore, the basin cells are precisely located and assigned to the reasonable large attraction criteria of attractors. Then the points studied are all iterated forward by numerical integration to locate their attractors. Thus, global analysis of systems is precisely carried out with satisfactory computational efficiency. However, to study a large region, integration steps for numerical integration must be carefully assigned to avoid large truncation error [6]; in addition, adjustable step sizes [7] can be used to improve computational efficiency. These two topics are not pursued in this paper. More detailed discussions can be found in references [6, 7].

4. PARAMETRIC ANALYSIS USING MICM

In the parametric analysis of a dynamical system, the attractors in the region of interest are located to understand creation, evolution and destruction of attractors at different values of parameters. When more points in a large region are studied, the parametric analysis of the system is more complete in a global sense. Here, parametric analysis is carried out by the method developed in the last section. The attracting cells of attractors in attractive regions are first located by MICM. The 3^n interior-and-boundary sampling points of each attracting cells are iterated forward by numerical integration to locate precise attractors, in which the criteria 10^{-6} and 10^{-4} are used to locate periodic attractors and their basins of attraction.

The proposed method is applied to the parametric analysis of a forced Duffing's system, that was also studied by ICM [8], governed by the following equation:

$$\ddot{x} + 0{\cdot}10\dot{x} - x + x^3 = 3{\cdot}2\cos(\omega t), \tag{2}$$

where $0{\cdot}473 \leqslant \omega \leqslant 0{\cdot}482$. In this analysis, the variation of ω is assigned to 0·0001 and the large region of study assigned to $(-10, 10)\times(-10, 10)$. Figure 2(a) and 2(b) show the positions and periods of the attractors located at different values of the forcing frequency, ω. When ω is varied from 0·482 to 0·475, the response experiences a period doubling cascade as found by ICM [8]. Then this system has two strange attractors at $\omega \leqslant 0{\cdot}4753$. These two strange attractors are further combined into a large strange attractor at $\omega \leqslant 0{\cdot}4736$. Figure 2(c) shows the two strange attractors at $\omega = 0{\cdot}4740$ located by the proposed method, and which are included in two attractive regions. The period of a strange attractor is indicated by 20 as in reference [8]. The two strange attractors are further combined into a large strange attractor at $\omega \leqslant 0{\cdot}4736$. Figure 2(d) shows the large strange attractor at $\omega = 0{\cdot}4730$. In addition, the computational time required for the parametric analysis of this system at each ω within [0·4736, 0·4820] is about 7 min, and about 2 h at each ω within [0·4730, 0·4735], within which most of the computational times required are used to locate exact strange attractors.

5. DETERMINATION OF FRACTAL-LIKE BASINS OF ATTRACTION

MICM is then applied to determine fractal-like basins of attraction of two-dimensional dynamical system. When MICM is applied to global analysis, some basin cells located may be defined by nine centers of cells residing in fractal regions. These basin cells are not

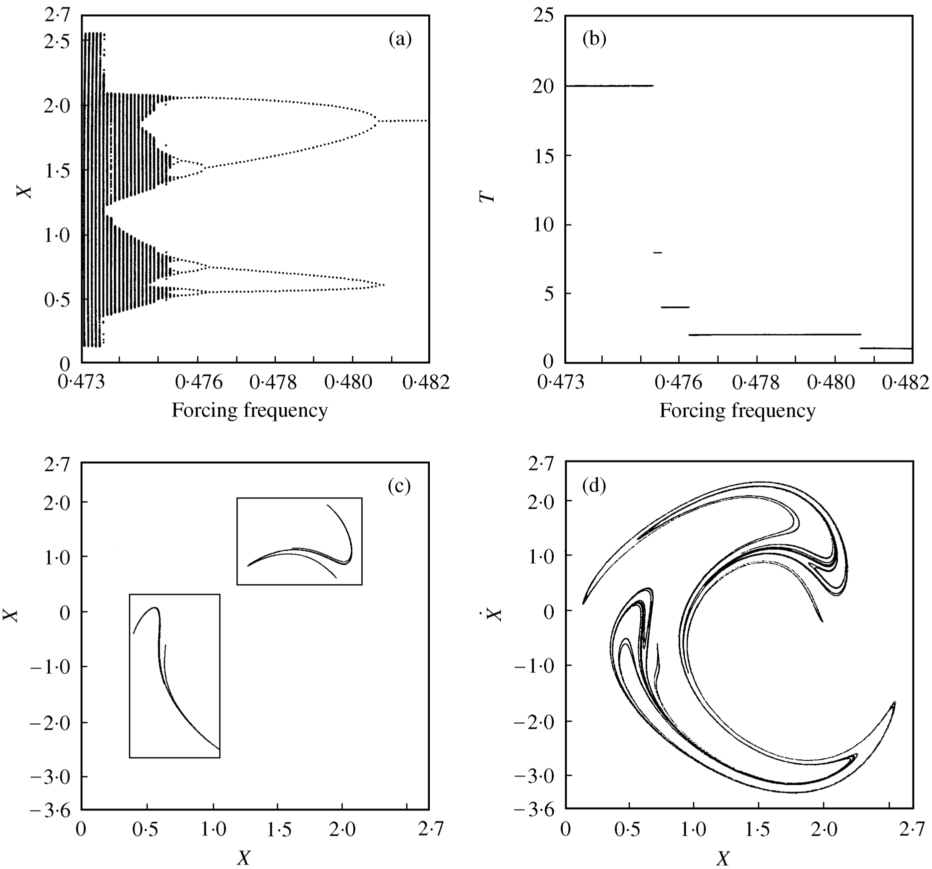

Figure 2. Parametric analysis of system (2) with a ω range of [0·473, 0·482], (a, b) positions and periods of the attractors at different values of ω (c) the two attractive regions, respectively, including a strange attractor at $\omega = 0{\cdot}4740$, (d) the combined strange attractor at $\omega = 0{\cdot}4730$.

appropriate attraction criteria of attractors. We will reduce the basins of attraction located by global analysis using MICM to positively invariant sets [3] of small cells under sample mappings or under interpolated mappings. Each positively invariant set is assigned to the attraction criterion of an attractor located by MICM.

As stated previously, high-resolution analysis of ICM is often used to determine fractal-like basins of attraction of dynamical systems. To study the correctness of such an analysis, we first apply ICM to the global analysis of a forced damped pendulum system, that was studied by the high-resolution analysis of ICM [8], governed by the following equation:

$$\ddot{\theta} + 0{\cdot}2\dot{\theta} + 1{\cdot}0\sin(\theta) = 2{\cdot}0\cos(t), \tag{3}$$

where the region of interest is $-\pi \leqslant \theta < \pi$, $-5{\cdot}0 \leqslant \dot{\theta} \leqslant 5{\cdot}0$. The system is further studied by high-resolution analysis of ICM with 101^2 cells used and 909^2 points studied. Figure 3(b–d) show the basins of attraction located by ICM in the region of interest and in the two fractal regions, $(0{\cdot}6\pi, \pi) \times (0{\cdot}0, 2{\cdot}0)$ and $(0{\cdot}86\pi, \pi) \times (0{\cdot}3, 1{\cdot}0)$. As compared with analysis using IGP, there are 61 920 (7·5%) points with attractors incorrectly located in the

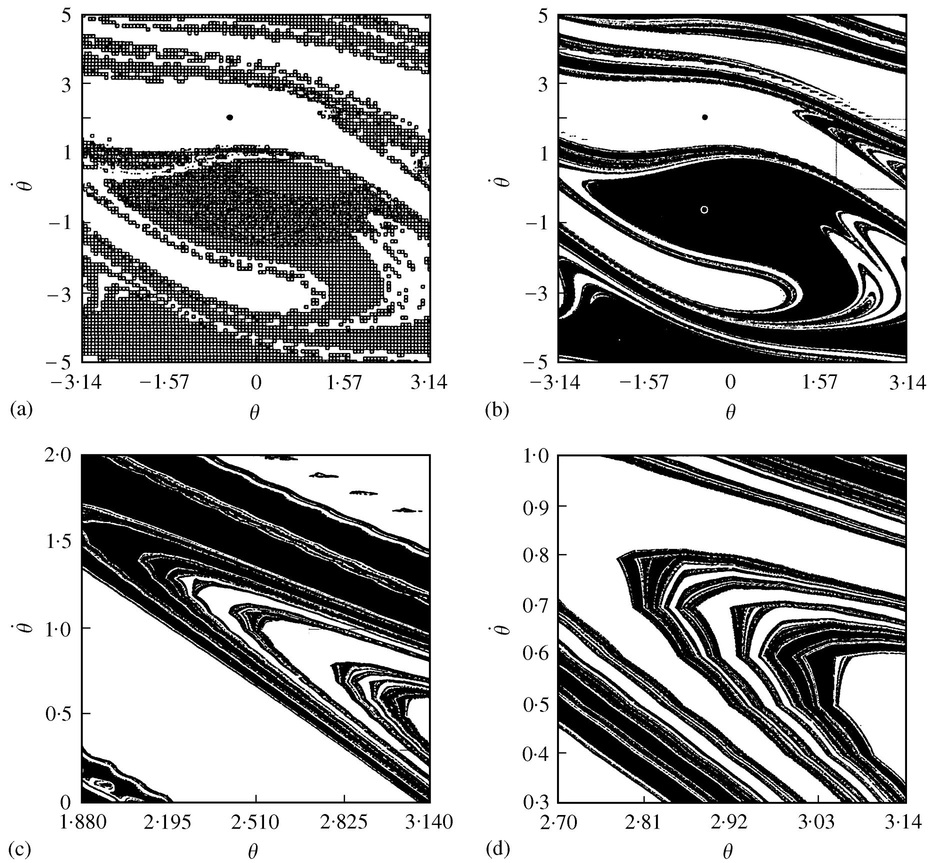

Figure 3. Basins of attraction of system (3) located by ICM with 101^2 cells used: (a) by global analysis, in the region of interest; (b) by high-resolution analysis, in the region of interest; (c, d) by high-resolution analysis, in the two fractal regions.

region of interest, and 105 224 (12·7%) and 172 817 (20·9%) points in the first and second fractal regions. Therefore, high-resolution analysis of ICM may not determine the fractal-like basins of attraction of a system correctly.

After global analysis using either MICM or ICM, the basins of attraction located can be used to preliminarily examine whether the basin boundaries of a system are fractal. If some cells in each basin of attraction are not situated continuously, such as those shows in Figure 3(a), the system possibly has fractal basin boundaries; otherwise, the system generally has smooth basin boundaries. If a system has fractal-like basins of attraction, the basins of attraction located by global analysis using MICM can be further used to distinguish fractal regions from continuous basins of attraction as follows. First, each cell is initially divided into four "small cells" that are assumed to be attracted by the attractor of the cell. Thus, each basin of attraction is indicated by a set of small cells; but some small cells may reside in fractal regions. Then, two algorithms are proposed to reduce the initial sets of small cells to positively invariant sets of small cells, that indicate continuous basins of attraction and are assigned to attraction criteria of attractors.

In the first algorithm, the mappings of four corners of a small cell after one period, in which one from numerical integration and three from interpolation, are assigned to the sample mappings of the small cell. If a sample mapping of a small cell leads to another basin of attraction, the small cell may be one in the fractal region. Thus, this small cell is no longer regarded as a small cell in the continuous basin of attraction. The reduction of the sets of small cells is continued until the sample mapping of each small cell resides in the same set of small cells. Eventually, each set is a positively invariant set under sample mappings of small cells.

In the second algorithm, more strict positively invariant sets under interpolated mappings are located to indicate continuous basins of attraction. The four sample mappings of a small cell form a quadrilateral. If the quadrilateral overlaps with other basins of attraction, the first interpolated mappings of some points in the small cell will reach other basins of attraction. Thus, the small cell may be one in the fractal regions, and is no longer considered to be a small cell in the continuous basin of attraction. According to this examination, eventually, the first interpolated mappings of all points in a small cell in a set always reside in the quadrilateral that is encompassed by the small cells in the same set. The sets of small cells finally obtained are positively invariant sets under interpolated mappings. This can be simply derived as follows. Let the set A include all the finally obtained small cells attracted by an attractor, and the set B include all the corresponding quadrilaterals. After the above examination, $B \subseteq A$. Under the interpolated mapping G, the first mapping of each point in the small cells resides in the quadrilaterals, and thus $G(A) = B$. Therefore, $G(A) \subseteq A$ and A is a positively invariant set under interpolated mappings. To examine whether a quadrilateral overlaps with other basins of attraction, some points can be sampled from the corresponding small cell and their first interpolated mappings are used to reduce the sets of small cells as in the first algorithm. In this paper, 4^2 interior-and-boundary points [3] are sampled from each small cell.

By using each algorithm, the finally obtained sets of small cells should correspond to most of the continuous basins of attraction and are assigned to the attraction criteria of attractors. Each point in a set is hence considered in the basin of attraction of corresponding attractor. However, the attractors of the points outside these sets of small cells are still unknown. To locate complete basins of attraction, these points are iterated forward by numerical integration until they are mapped to the positively invariant sets to identify their attractors. Thus, global analysis using this method is complete since the attractors of all points are located. The proposed method is more computationally efficient than IGP since a smaller region is studied and larger attraction criteria of attractors are used. It can also be estimated that larger positively invariant sets can be obtained as more cells are used by global analysis using MICM. This is due to the fact that the sample mappings of a small cell must all reside in a set to keep the small cell in the set. This examination is strict. More cells with smaller cell size can reduce the strictness of examination.

The proposed method is used here to determine fractal-like basins of attraction of dynamical systems. System (3) is studied with 303^2 cells used and 909^2 points studied. Figure 4(a) shows the basins of attraction located in the region of interest. Figure 4(b) shows the two positively invariant sets of small cells under sample mappings, and which occupy 73·3% of the region of interest. Figure 4(c) and 4(d) show the basins of attraction located in the two fractal regions, $(0{\cdot}6\,\pi,\ \pi) \times (0{\cdot}0,\ 2{\cdot}0)$ and $(0{\cdot}86\,\pi,\ \pi) \times (0{\cdot}3,\ 0{\cdot}1)$. When compared with analysis using IGP, there are 742 points in the region of interest with attractors incorrectly located; 400 and 321 points in the fractal regions 1 and 2. The basins of attraction located are far more precise than those located by ICM. Besides, the errors of the basins of attraction located do not increase in the fractal regions as in ICM analysis. If the final sets of small cells are alternatively located as positively invariant sets under interpolated mappings, the basins of attraction located in the three regions are all the same as those from IGP. The precision of the basins of attraction located is further improved with more computational times required, as listed in Tables 1 and 2.

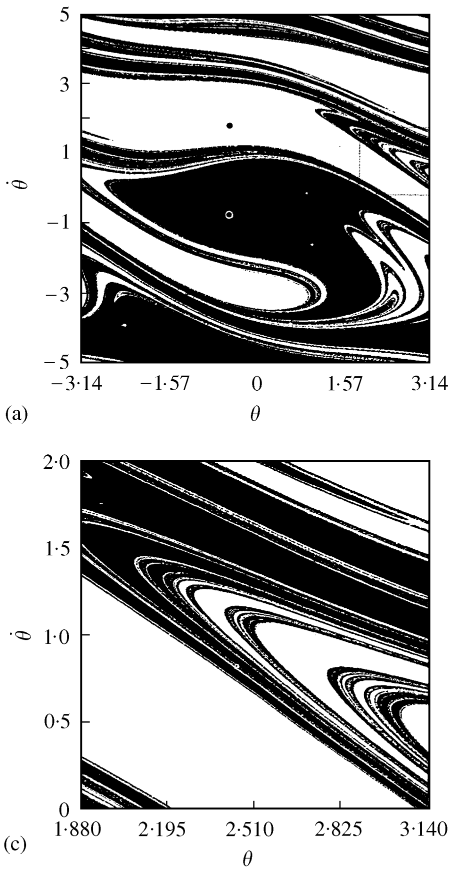

Figure 4. Studies of system (3) using MICM with 303^2 cells: (a) two attractors and basins of attraction located in the region of interest, (b) two positively invariant sets under sample mappings of small cells, (c, d) basins of attraction located in the two fractal regions.

TABLE 1

Computational times (s) required for various analyses of the regions of interest

Computational method	# of cells	PIS area	Average periods	Computational Time	Computational Efficiency	# of points of error	Studied system
ICM	101^2			1169	68·2	61 920	(3)
ICM	303^2			1555	51·3	29 921	(3)
MICM‡	303^2	57·6%	2·61	10 096	7·9	0	(3)
MICM†	303^2	73·3%	0·86	3337	23·9	742	(3)
IGP			21·93	79 715	1·0	0	(3)
ICM	303^2			1141	52·2	159 871	(4)
MICM‡	303^2	44·6%	2·55	7238	8·2	378	(4)
IGP			21·80	59 543	1·0	0	(4)
ICM	303^2			1208	52·0	53 779	(5)
MICM‡	303^2	59·5%	2·03	5784	10·9	19	(5)
IGP			23·19	62 798	1·0	0	(5)

TABLE 2

Computational times (s) required for various analyses of fractal regions

Computational method	# of cells	PIS area	Average periods	Computational Time	Computational Efficiency	# of points of error	Studied system
ICM	101^2			1178	67·2	105 224	(3), f1
ICM	303^2			1551	50·5	56 021	(3), f1
MICM‡	303^2	57·6%	3·06	11 751	6·7	0	(3), f1
MICM†	303^2	73·3%	1·41	5394	14·6	400	(3), f1
IGP			21·66	78 600	1·0	0	(3), f1
ICM	101^2			1176	74·3	172 871	(3), f2
ICM	303^2			1545	55·8	89 550	(3), f2
MICM‡	303^2	57·6%	5·27	20 535	4·2	0	(3), f2
MICM†	303^2	73·3%	2·76	10 418	8·3	321	(3), f2
IGP			23·77	86 823	1·0	0	(3), f2
ICM	101^2			929	72·9	178 478	(5), f1
ICM	303^2			1210	56·0	109 726	(5), f1
MICM‡	303^2	59·5%	3·36	9351	7·2	18	(5), f1
IGP			24·97	67 752	1·0	0	(5), f1

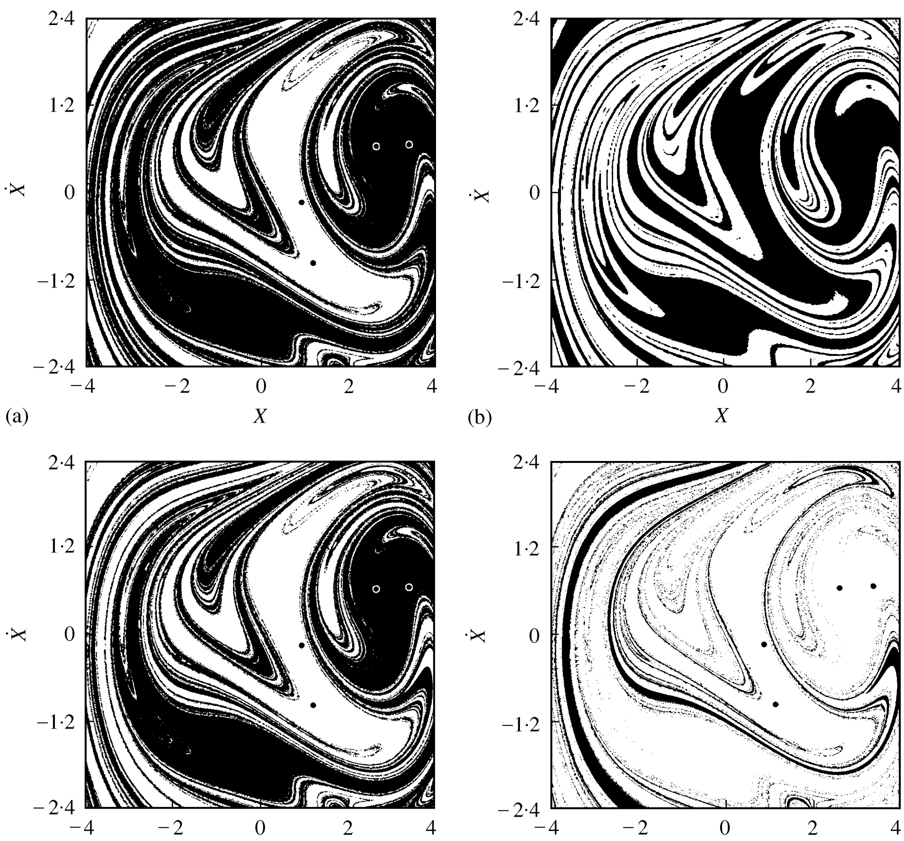

Figure 5. Studies of system (4) using MICM with 303^2 cells: (a) two attractors and basins of attraction located in the region of interest, (b) two positively invariant sets under interpolated mappings, (c, d) basins of attraction of attractor 1 and the sink cell with the assumption of sink cell preserved.

The proposed method is then applied to the studies of another Duffing's system, that was also studied by ICM [9], governed by the following equation:

$$\ddot{x} + 0{\cdot}10\dot{x} - x + x^3 = 3{\cdot}2\cos(0{\cdot}4776t). \tag{4}$$

In this analysis, 303^2 cells are used and 909^2 points are studied in the region of interest $(-2{\cdot}4, 2{\cdot}4)\times(-4{\cdot}0, 4{\cdot}0)$. Two periodic attractors of period two are located by global analysis using MICM, with the basins of attraction 1 and 2 shown in Figure 5(a) as black and blank areas. There are 378 points with attractors incorrectly located. Figure 5(b) indicates the two positively invariant sets under interpolated mappings. If the assumption of the sink cell is preserved, the basins of attraction of attractor 1 and the sink cell are shown in Figure 5(c) and 5(d).

The proposed method is finally applied to the studies of another Duffing's system, that was also studied by numerical integration [10], governed by the following equation:

$$\ddot{x} + 0{\cdot}168\dot{x} - 0{\cdot}5x + 0{\cdot}5x^3 = 0{\cdot}15\sin(t). \tag{5}$$

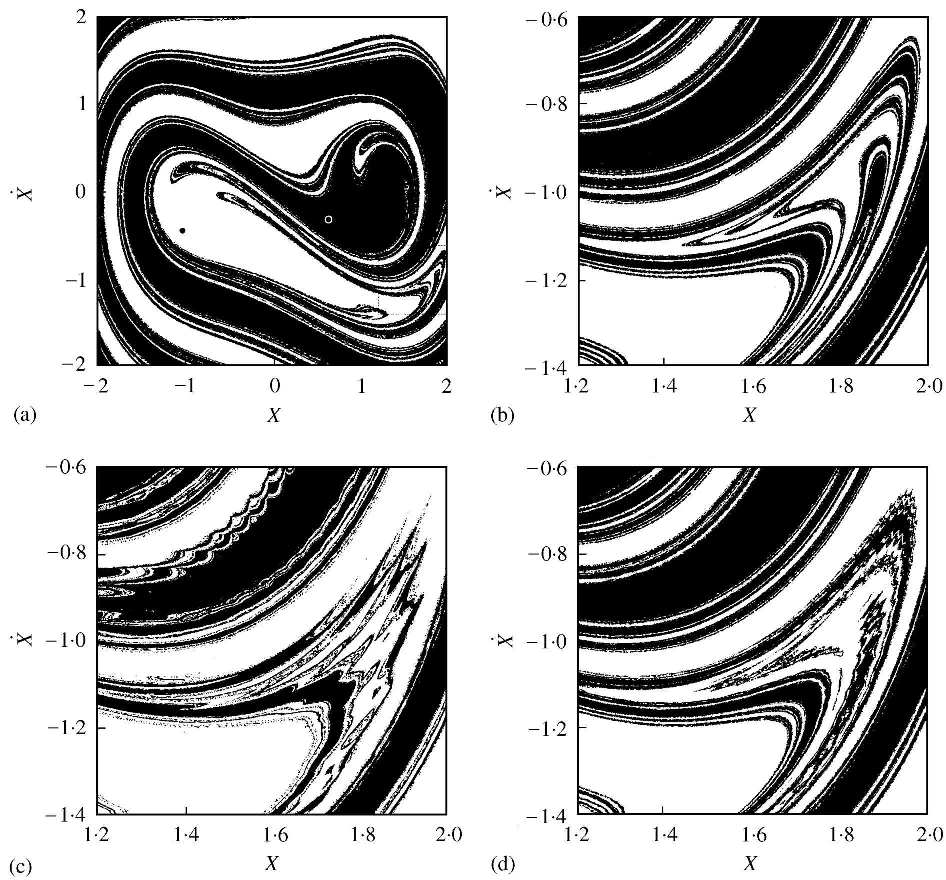

Figure 6. Studies of system (5): (a) two attractors and basins of attraction in the region of interest located by MICM with 303^2 cells used, (b) basins of attraction in the fractal region located by MICM with 303^2 cells used, (c, d) basins of attraction in the fractal region located by ICM with 101^2 and 303^2 cells used.

In this analysis, 303^2 cells are used and 909^2 points are studied in the region of interest $(-2{\cdot}0, 2{\cdot}0)\times(-2{\cdot}0, 2{\cdot}0)$. Two periodic attractors of period one are located by global analysis using MICM, and their basins of attraction are reduced to positively invariant sets under interpolated mappings. Figure 6(a, b) show the basins of attraction located by the proposed method in the region of interest and in the fractal region $(1{\cdot}2, 2{\cdot}0)\times(-1{\cdot}4, -0{\cdot}6)$, in which there are 19 and 18 points with attractors incorrectly located. The fractal region is also studied by high-resolution analysis of ICM with 101^2 and 303^2 cells used. The basins of attraction located are as shown in Figure 6(c, d), with the attractors of 178 478 and 109 726 points incorrectly located. The precision of analysis of fractal regions is much improved by the proposed method over that using ICM.

The computational times required for different analyses are listed in Tables 1 and 2 for the studies of the regions of interest and fractal regions. "MICM‡" and "MICM†" indicate the MICM methods assigning positively invariant sets of small cells under interpolated mappings and under sample mappings to attraction criteria of attractors, "PIS area" for the area ratio of the located positively invariant sets to the region of interest. '# of cells" indicates the number of cells used in the global analysis using a cell mapping method. All analyses study 909^2 points uniformly sampled in the region of interest. "# of points of error" indicates the number of points with attractors incorrectly located. "Average periods" indicate the average periods required by all points to identify their attractors. "Computational efficiency" indicates the computational improvement of a method over IGP. Both IGP and ICM use the criteria 10^{-6} and 10^{-4} to locate periodic attractors and their basins of attraction. In the global analysis using IGP, each mapping of a point after construction is immediately taken to verify whether this point is attracted by each of the periodic attractors located. In the global analysis using ICM, interpolated mappings of each point are constructed within 100 periods to examine whether the point is attracted by a strange attractor.

6. CONCLUDING REMARKS

In this paper, a method using MICM is developed to locate all attractors of a dynamical system in a large region. Since the large region can be arbitrarily assigned, this method is quite helpful for the studies of the systems that were never studied before. The basins of attraction located by this method are the same as those from IGP with improved computational efficiency for the case studied in this paper. The parametric analysis of a dynamical system is then studied by this method. This parametric analysis is more precise than that using ICM. Furthermore, fractal-like basins of attraction are determined by MICM with positively invariant sets under sample mappings or under interpolated mappings assigned to attraction criteria of attractors. The fractal-like basins of attraction located by the proposed method are almost the same as those located by IGP, with a 10-fold improvement in computational efficiency. The methods in this paper were developed to improve the computational efficiency over IGP with minimal completeness and accuracy sacrificed in the analysis.

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REFERENCES

1. J. M. T. Thompson and H. B. Stewart 1986 *Nonlinear Dynamics and Chaos: Geometric Method for Engineers and Scientists*. Chichester: John Wiley and Sons.
2. Z.-M. Ge and S.-C. Lee 1997 *Journal of Sound and Vibration* **199**, 189–206. A modified interpolated cell mapping method.
3. C. S. Hsu 1987 *Cell-to-Cell Mapping: A Method of Global Analysis for Nonlinear Systems*. New York: Springer-Verlag.
4. B. H. Tongue and K. Gu 1988 *American Society of Mechanical Engineers Journal of Applied Mechanics* **55**, 461–466. Interpolated cell mapping of dynamical systems.
5. C. A. Hsu and H. M. Chiu 1987 *Journal of Sound and Vibration* **114**, 203–218. Global analysis of a system with multiple responses including a strange attractor.
6. L. G. Ixaru 1992 *Numerical Methods for Differential Equations and Applications*. Dordrecht: D. Reidel Publishing Company.
7. G. E. Forsythe, M. A. Malcolm and C. B. Moler 1977 *Computer Methods for Mathematical Computations*. Englewood Cliffs, NJ: Prentice-Hall.
8. B. H. Tongue 1987 *Physica D* **28**, 401–408. On obtaining global nonlinear system characteristics through interpolated cell mapping.
9. B. H. Tongue and K. Gu 1988 *SIAA Journal of Applied Mathematics* **48**, 1206–1214. A theoretical basis for interpolated cell mapping.
10. C. Pezeshki and E. H. Dowell 1987 *Journal of Sound and Vibration* **117**, 219–232. An examination of initial condition maps for the sinusoidally excited buckled beam modeled by the Duffing's equation.